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DISCUSSION OF: FINITE ELASTO-PLASTIC DEFORMATION—I. THEORY AND NUMERICAL EXAMPLES[1]

J. P. CARTER
Department of Civil Engineering, King's College, London, England

The authors have presented a numerical method for solving problems involving finite elasto-plastic deformations. Their formulation is based on a rate approach with a constitutive law

$$\hat{\sigma}^{ij} = P^{ijkl} d_{kl} \tag{1}$$

where

$$\hat{\sigma}^{ij} \equiv \dot{\sigma}^{ij} + \sigma_m{}^i w^{mj} - \sigma_m{}^j w^{im}\dagger \tag{2}$$

is the Jaumann stress rate;

$$d_{ij} = (1/2)(v_{i;j} + v_{j;i}) \tag{3}$$

and

$$w_{ij} = (1/2)(v_{i;j} - v_{j;i}) \tag{4}$$

are the deformation rate and spin tensors respectively and P^{ijkl} is a matrix of values which are functions of the current stress state. With the material law of eqn (1), the authors have obtained both a finite element and an analytical solution to the problem of the finite extension of a cube of elastic material in plane strain and plane stress.

The writer has also obtained such a solution for the plane strain example (Fig. 1). Two forms of Hooke's Law were investigated. These are, in plane strain terms:

(i) If during an increment in loading a material particle displaces from a point P_i in space to a point P_{i+1}, then denote the initial total stress and total strain at P_i by the vectors $\boldsymbol{\sigma}^{(i)} = (\sigma_{xx}^{(i)}, \sigma_{yy}^{(i)}, \sigma_{xy}^{(i)})^T$ and $\mathbf{e}^{(i)} = (e_{xx}^{(i)}, e_{yy}^{(i)}, e_{xy}^{(i)})^T$ respectively. Similarly at point P_{i+1} after the load increment is applied denote the total stress and strain by the vectors $\boldsymbol{\sigma}^{(i+1)} = (\sigma_{xx}^{(i+1)}, \sigma_{yy}^{(i+1)}, \sigma_{xy}^{(i+1)})^T$ and $\mathbf{e}^{(i+1)} = (e_{xx}^{(i+1)}, e_{yy}^{(i+1)}, e_{xy}^{(i+1)})^T$ respectively. We postulate Hooke's law, in this case, to be

$$\dot{\boldsymbol{\sigma}}^{(i+1)} = D\dot{\mathbf{e}}^{(i+1)}. \tag{5}$$

Defining quantities $\Delta\boldsymbol{\sigma}$ and $\Delta\mathbf{e}$ by

$$\Delta\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(i+1)} - \boldsymbol{\sigma}^{(i)} \tag{6}$$

$$\Delta\mathbf{e} = \mathbf{e}^{(i+1)} - \mathbf{e}^{(i)} \tag{7}$$

and noting that $\dot{\boldsymbol{\sigma}}^{(i)} = \dot{\mathbf{e}}^{(i)} = \mathbf{0}$ for this particular load increment, then (5) reduces to

$$\Delta\dot{\boldsymbol{\sigma}} = D\Delta\dot{\mathbf{e}}. \tag{8}$$

The matrix D is assumed to be given by

$$D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix}$$

(E = Young's modulus and ν = Poisson's ratio)

†The symbol "˙" denotes material differentiation.

as in the infinitesimal case, and the strain vector $\Delta\mathbf{e}$ consists of the Lagrange components for this increment. Note that $\boldsymbol{\sigma}^{(i)}$ and $\mathbf{e}^{(i)}$ are formed merely by the summation of all previous incremental values.

This form of constitutive law is equivalent to that of the authors (eqn 1) if we consider the products of initial stress components and rotation to be negligible. This may be achieved by using small loading increments, and indeed, if rotation is absent, then eqn (1) and eqn (5) are equivalent.

(ii) The second adopted form of Hooke's law was suggested by Biot[2]. It is

$$t = C\Delta\mathbf{e} \tag{10}$$

where $\mathbf{t} = (t_{11}, t_{22}, t_{12})^T$, C is a symmetric matrix of material constants and $\Delta\mathbf{e}$ is as above. The vector $\mathbf{t}$ was considered by Biot to be the true increment of stress for an increment of deformation in an initially stressed medium. The quantities $\Delta\boldsymbol{\sigma}$ and $\mathbf{t}$ are related by

$$\begin{aligned} t_{11} &= \Delta\sigma_{xx} + \sigma_{xx}^{(i)}\Delta e_{yy} - \sigma_{xy}^{(i)}(\Delta e_{xy} - 2w) \\ t_{22} &= \Delta\sigma_{yy} + \sigma_{yy}^{(i)}\Delta e_{xx} - \sigma_{xy}^{(i)}(\Delta e_{xy} + 2w) \\ t_{12} &= \Delta\sigma_{xy} + \tfrac{1}{2}\sigma_{xy}^{(i)}(\Delta e_{xx} + \Delta e_{yy}) - \tfrac{1}{2}\sigma_{xx}^{(i)}(\Delta e_{xy} + w) - \tfrac{1}{2}\sigma_{yy}^{(i)}(\Delta\sigma_{xy} - w) \end{aligned} \tag{11}$$

where w is the rigid body rotation associated with the deformation. For the numerical example the writer has taken $C = D$.

Figure 1 shows a comparison of the load (P)—stretch (λ) response of the elastic material for each form of Hooke's law. Poisson's ratio was taken as 0·4. The writer's results were obtained using both a finite element technique and a direct solution. Both agreed to within 3%. Table 1

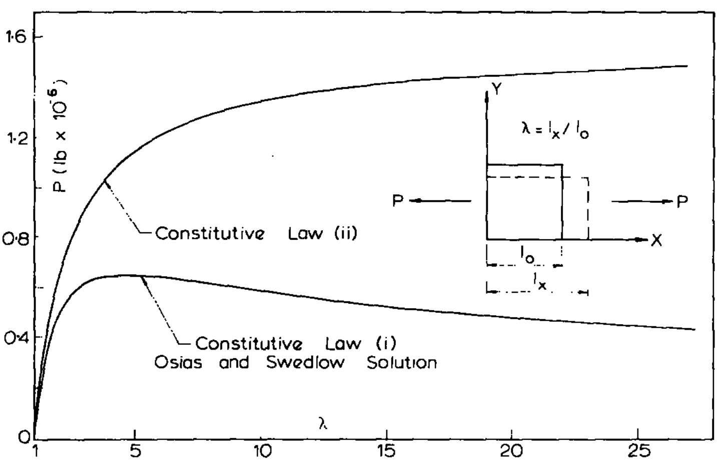

Fig. 1. Unit cube extension.

Table 1. Comparison of solutions for the homogeneous extension of elastic† bodies

	P_c lb × 10^{-5}		λ_c		σ_c lb/in² × 10^{-6}	
	Exact	Finite element	Exact	Finite element	Exact	Finite element
Osias and Swedlow	6·57	6·59	4·48	4·46	1·79	1·79
Hooke's Law Form (i)	6·57‡	6·57	4·48¶	4·54	1·79	1·80
Hooke's Law Form (ii)	17·86§	—	∞	—	∞	—

†$E = 10^6$ lb/in², $\nu = 0{\cdot}4$ in all cases and in general.
‡$P_c = 0{\cdot}36788E/(\nu^2 + \nu)$.
§$P_c = E/(\nu^2 + \nu)$.
¶$\lambda_c = e^{(1-\nu)/\nu}$.
$l_0 = 1{\cdot}0$.

shows a comparison of the maximum load P_c and the associated values λ_c and σ_c, for each method of analysis. Note in particular that eqn (1) and eqn (8) lead to identical solutions in this trivial case involving zero-rotation. But these results also show, by simple example, how the solution to a problem involving finite deformations can be very different, depending on the form adopted for the constitutive law.

REFERENCES

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2. M. A. Biot, *The Mechanics of Incremental Deformations*. Wiley, New York (1965).